TRIANGULAR NUMBERS—SOME GENERAL THEOREMS AND RELATED RESULTS

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Dedicated to my esteemed teacher, the inimitable N. B. N. – SSKZM

Abstract. The following results are proved.
If $\mathbf{T} = \{T_0, T_1, T_2, T_3, \text{---}T_n, \text{---}, \infty\} \quad [T_0 = 0]$ are Triangular numbers

1. then $(2\xi + 1)^2 T_n + T_\xi \quad [\xi \geq 0, n \geq 0]$ are also Triangular Numbers.
2. then $(2\lambda)^2 T_n + T_\lambda + \lambda n [\lambda \geq 1, n \geq 0]$ are also Triangular numbers.
3. The square of every odd number can be expressed as the difference of two Triangular Numbers.
4. The square of every λ^{th} even number can be expressed as the difference of the '$(3\lambda)^{\text{th}}$ Triangular Number' and 'the sum of the λ^{th} Triangular Number and λ'.

5. For any Natural Number $N \geq 2$, there Exist Infinite Triangular Numbers that are the Sum of N Triangular Numbers each.

6. If

$$\mathbf{x} = \frac{n(n+2\alpha+1)}{2} + 1,\ y = n+1,\ z = \frac{n(n+2\alpha+1)}{2}$$

$$\mathbf{T_x} = T_y + T_z + (\alpha-1)n = T_y + T_z + (\alpha-1)(y-1)$$

$$[\alpha = 0, 1, 2, \text{---}\infty]\ [n = 0, 1, 2, \text{---}, \infty]$$

7. If

$$\mathbf{x} = \frac{(n+1)(n+2\beta)}{2} + 1,\ y = n+1,\ z = \frac{(n+1)(n+2\beta)}{2}$$

$$\mathbf{T_x} = T_y\ +\ T_z + \beta + (\beta-1)n = T_y + T_z + \beta$$

$$+ (\beta-1)(y-1)[\beta = 1, 2, 3\text{---}\infty]\ [n = 0, 1, 2, \text{---}, \infty]$$

Various related sub-results are also proved.

Infinite Types of unique sub-classes of Triangular Numbers are defined and the formulae or their sum upto "n" terms are derived and these are expressed in terms of binomial coefficients also.

Key words:- Triangular Numbers, Binomial Coefficiets, Infinite Generalizations.

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1 Some General Results

Theorem 1.1. *If* $\mathbf{T} = \{T_0, T_1, T_2, T_3, \text{---}T_n, \text{---}\infty\}[T_0 = 0]$ *are Triangular Numbers then* $(2\xi+1)^2 T_n + T_\xi$ $[\xi \geq 0, n \geq 0]$ *are also Triangular Numbers.*

[Euler's results that $(9T_n + 1), (25T_n + 3), (49T_n + 6), [n \geq 0]$ *are Triangular, are obtained as special cases when we set* $\xi = 1, 2,$ *and 3 respectively in the formula* $(2\xi + 1)^2 T_n + T_\xi$*.]*

Proof. We prove the result by repeated induction.

For $\xi = 0, (2\xi+1)^2 T_n + T_\xi$ are the set of Triangular Numbers themselves. For $\xi = 1, 2, 3$ we have the standard results due to Euler which are proved by induction on "n" for each specific value of ξ. To prove the general formula for all values of $\xi \geq 0$ we assume that the result is true for $\xi = k$ and prove that it is true for $\xi = k+1$ and thus close the doubly (infinitely) inductive argument. We state the complete set of Triangular Numbers for each value of ξ as follows which can easily be proved by induction and the Pythagorean. Theorem that a number is Triangular if and only if it is of the form $\frac{n(n+1)}{2}$, in each case.

[Note on the notation:
In $\overset{[R]}{{}^{L}T_n}$, R may be called the Root Level and L the Layer level which define a specific sub-class of Triangular Numbers.]

For $\xi = 0, 2\xi + 1 = 1$

$$\overset{[\mathbf{1}]}{{}^{\mathbf{1}}\mathbf{T_n}} = \frac{n(n+1)}{2} = T_n + T_0 = (T_0 + T_1)T_n + T_0 = T_n \quad [n \geq 1]$$

For $\xi = 1, 2\xi + 1 = 3$

$$\mathbf{{}^{[3]}_{1}T_n} = \frac{(3n+1)(3n+2)}{2} = 9T_n + T_1 = (T_2 + T_3)T_n + T_1$$
$$= T_{(3n+1)} \quad [n \geq 1]$$

For $\xi = 2, 2\xi + 1 = 5$

$$\mathbf{{}^{[3]}_{2}T_n} = \frac{(6n+1)(6n+2)}{2} = 25T_n + T_2 = (T_4 + T_5)T_n + T_2$$
$$= T_{(6n+1)} \quad [n \geq 1]$$

For $\xi = 3, 2\xi + 1 = 7$

$$\mathbf{{}^{[3]}_{3}T_n} = \frac{(9n+1)(9n+2)}{2} = 49T_n + T_3 = (T_6 + T_7)T_n + T_3$$
$$= T_{(9n+1)} \quad [n \geq 1]$$

For $\xi = 4, 2\xi + 1 = 9$

$$\mathbf{{}^{[3]}_{4}T_n} = \frac{(12n+1)(12n+2)}{2} = 81T_n + T_4$$
$$= (T_8 + T_9)T_n + T_4 = T_{(12n+1)} \quad [n \geq 1]$$

For $\xi = 5, 2\xi + 1 = 11$

$$\mathbf{{}^{[3]}_{5}T_n} = \frac{(15n+1)(15n+2)}{2} = 121T_n + T_5$$
$$= (T_{10} + T_{11})T_n + T_5 = T_{(15n+1)} \quad [n \geq 1]$$

For $\xi = 6, 2\xi + 1 = 13$

$$\overset{[\mathbf{3}]}{{}^{\mathbf{6}}\mathbf{T_n}} = \frac{(18n+1)(18n+2)}{2} = 169T_n + T_6$$
$$= (T_{12} + T_{13})T_n + T_6 = T_{(18n+1)} \qquad [n \geq 1]$$

Clearly if $\xi = k$

$$\overset{[\mathbf{3}]}{{}^{\mathbf{k}}\mathbf{T_n}} = \frac{(3kn+1)(3kn+2)}{2} = (2k+1)^2 T_n + T_k$$
$$= (T_{2k} + T_{2k+1}) + T_n + T_k = T_{(3kn+1)} \quad [n \geq 1])$$

are Triangular, then for $\xi = k + 1$

$$\overset{[\mathbf{3}]}{\mathbf{k+1}\mathbf{T_n}} = \frac{(3(k+1)n+1)(3(k+1)n+2)}{2}$$
$$= (2(k+1)+1)^2 T_n + T_{k+1}$$
$$= (T_{2(k+1)} + T_{2(k+1)+1}) + T_n + T_{k+1}$$
$$= T_{(3(k+1)n+1)} \quad [n \geq 1]$$

are also Triangular thus proving that

$$\overset{[\mathbf{3}]}{\boldsymbol{\xi}\mathbf{T_n}} = \frac{(3\xi n+1)(3\xi n+2)}{2} = (2\xi+1)^2 T_n + T_\xi$$
$$= (T_{2\xi} + T_{2\xi+1})T_n + T_\xi = T_{(3\xi n+1)} \qquad [\xi \geq 1, n \geq 1] \qquad (1.1)$$

are all Triangular. Thus we may say that Triangular Numbers derived form Root Level 3 Layer Levels $\xi[\xi \geq 1]$ are Eulerian classes of Triangular Numbers and we may call "ξ" the Eulerian Class Level Number. For $\xi = 0$ the Eulerian class is the set of all Triangular Numbers. □

Corollary 1.1. *Every Triangular Number can be Expressed the Difference of a Triangular Number and the Sum of two Triangular numbers and the product of another Triangular Number in Infinite Ways.*

$$\mathbf{T}_{\boldsymbol{\xi}} = T_{(3\xi n+1)} - (T_{2\xi} + T_{2\xi+1})T_n \qquad [\xi \geq 1, n \geq 1]$$

Proof. Trivial from (1.1). □

Corollary 1.2. *Every Triangular Number can be Expressed as the Ratio of the Difference of Two Triangular Numbers and the Sum of Two Triangular Numbers in Infinite Ways.*

$$\mathbf{T}_{\mathbf{n}} = \frac{T_{(3\xi n+1)} - T_{\xi}}{(T_{2\xi} + T_{2\xi+1})} \qquad [\xi \geq 1, n \geq 1]$$

Proof. Trivial form (1.1). □

Corollary 1.3. *For* $\xi \geq 1, n \geq 1$

$$(\mathbf{2}\boldsymbol{\xi} + \mathbf{1})^{\mathbf{2}}(\mathbf{n} + \mathbf{1})^{\mathbf{2}} = T_{(3\xi n+1)} + T_{(3\xi(n+1)+1)} - 2T_{\xi}$$

Proof.

$$\begin{aligned} \overset{[\mathbf{3}]}{\boldsymbol{\xi}}_{\mathbf{T_n}} &= \frac{(3\xi n+1)(3\xi n+2)}{2} = (2\xi+1)^2 T_n + T_{\xi} \\ &= (T_{2\xi} + T_{2\xi+1})T_n + T_{\xi} = T_{(3\xi n+1)} \quad [\xi \geq 1, n \geq 1] \qquad (1.2) \end{aligned}$$

$$\mathbf{(2\xi+1)^2T_n + T_\xi} = T_{(3\xi n+1)}. \tag{1.3}$$

Substituting $n = n+1$ in (1.3)

$$\mathbf{(2\xi+1)^2T_{n+1} + T_\xi} = T_{(3\xi(n+1)+1)} \tag{1.4}$$

adding the two equations, substituting $T_n + T_{n+1} = (n+1)^2$ [Nicomachus] and simplifying we have

$$\mathbf{(2\xi+1)^2(n+1)^2} = T_{(3\xi n+1)} + T_{(3\xi(n+1)+1)} - 2T_\xi$$
$$[\xi \geq 1, n \geq 1] \tag{1.5}$$

$2\xi + 1$ defines the set of all odd numbers. Seeking a symmetric result for all even numbers, we have the following theorem. □

Theorem 1.2. *If* $T = \{T_0, T_1, T_2, T_3, \text{---}T_n, \text{---}\infty\}[T_0 = 0]$ *are Triangular Numbers then* $(2\lambda)^2T_n + T_\lambda + \lambda n$ $[\lambda \geq 1, n \geq 0]$ *are also Triangular Numbers.*

Proof. We prove the result by repeated induction.

First we prove for $\lambda = 1, 2, 3$ by induction on "n" for each specific value of λ. To prove the general formula for all values of $\lambda \geq 1$ we assume that the result is true for $\lambda = k$ and prove that it is true for $\lambda = k+1$ and thus close the doubly (infinitely) inductive argument. We state the complete set of Triangular Numbers for each value of λ as follows which can easily be proved by induction and the Pythagorean Theorem that a number is Triangular if and only if if it is the form $\dfrac{n(n+1)}{2}$, in each case.

For $\lambda = 1,\ 2\lambda = 2$

$$\mathbf{2}^{[\mathbf{1}]}\mathbf{T_n} = \frac{(2n+1)(2n+2)}{2} = 4T_n + T_1 + 1n$$
$$= (T_1 + T_2)T_n + T_1 + 1(n) = T_{3n} \quad [n \geq 1]$$

For $\lambda = 2,\ 2\lambda = 4$

$$\mathbf{5}^{[\mathbf{1}]}\mathbf{T_n} = \frac{(5n+1)(5n+2)}{2} = 16T_n + T_2 + 2n$$
$$= (T_3 + T_4)T_n + T_2 + 2n = T_{6n} \quad [n \geq 1]$$

For $\lambda = 3,\ 2\lambda = 6$

$$\mathbf{8}^{[\mathbf{1}]}\mathbf{T_n} = \frac{(8n+1)(8n+2)}{2} = 36T_n + T_3 + 3n$$
$$= (T_5 + T_6)T_n + T_3 + 3n = T_{9n} \quad [n \geq 1]$$

For $\lambda = 4,\ 2\lambda = 8$

$$\mathbf{11}^{[\mathbf{1}]}\mathbf{T_n} = \frac{(11n+1)(11n+2)}{2} = 64T_n + T_4 + 4n$$
$$= (T_7 + T_8)T_n + T_4 + 4n = T_{12n} \quad [n \geq 1]$$

For $\lambda = 5,\ 2\lambda = 10$

$$\mathbf{14}^{[\mathbf{1}]}\mathbf{T_n} = \frac{(14n+1)(14n+2)}{2} = 100T_n + T_5 + 5n$$
$$= (T_9 + T_{10})T_n + T_5 + 5n = T_{15n}$$

For $\lambda = 6,\ 2\lambda = 12$

$$\mathbf{17}\overset{[\mathbf{1}]}{\mathbf{T_n}} = \frac{(17n+1)(17n+2)}{2} = 144T_n + T_6 + 6n$$
$$= (T_{11} + T_{12})T_n + T_6 + 6n = T_{18n} \quad [n \geq 1]$$

For $\lambda = 7,\ 2\lambda = 14$

$$\mathbf{20}\overset{[\mathbf{1}]}{\mathbf{T_n}} = \frac{(20n+1)(20n+2)}{2} = 196T_n + T_7 + 7n$$
$$= (T_{13} + T_{14})T_n + T_7 + 7n = T_{21n} \quad [n \geq 1]$$

For $\lambda = k,\ 2\lambda = 2k$

$$(\mathbf{3k-1})\overset{[\mathbf{1}]}{\mathbf{T_n}} = \frac{((3k-1)n+1)((3k-1)n+2)}{2}$$
$$= (2k)^2T_n + T_k + kn$$
$$= (T_{k-1} + T_k)T_n + T_k + kn = T_{3kn} \quad [n \geq 1]$$

For $\lambda = k+1,\ 2\lambda = 2(k+1)$

$$(\mathbf{3k+2})\overset{[\mathbf{1}]}{\mathbf{T_n}} = \frac{((3k+2)n+1)((3k+2)n+2)}{2}$$
$$= (2(k+1))^2T_n + T_{(k+1)} + (k+1)n$$
$$= (T_k + T_{k+1})T_n + T_{(k+1)} + (k+1)n = T_{(3k+1)n} \quad [n \geq 1]$$

are also triangular thus proving that

$$\mathbf{(3\lambda-1)}\overset{[\mathbf{1}]}{\mathbf{T_n}} = \frac{((3\lambda-1)n+1)((3\lambda-1)n+2)}{2} \\ = (2\lambda)^2 T_n + T_\lambda + \lambda n \\ = (T_{2\lambda-1} + T_{2\lambda})T_n + T_\lambda + \lambda n = T_{3\lambda n} \quad [n \geq 1] \tag{1.6}$$

are all Triangular. Thus we may say that Triangular Numbers derived from Root Level 1 Layer Levels $(3\lambda-1)[\lambda \geq 1]$ are Pythagorean classes of Triangular Numbers and we may call "$(3\lambda-1)$" the Pythagorean Class Level Number. □

Corollary 1.4. *Every Triangular Number can be Expressed the "Difference of a Triangular Number" and "the Sum of Two Triangular Numbers and the product of another Triangular Number and a specific constant (product of the two indexing factors)" in Infinite Ways.*

$$\mathbf{T_\lambda} = T_{3\lambda n} - (T_{2\lambda-1} + T_{2\lambda})T_n - \lambda n \qquad [\lambda \geq 1, n \geq 1]$$

Proof. Trivial from (1.6) □

Corollary 1.5. *Every Triangular Number can be Expressed as the Ratio of the "Difference of a Triangular Number and the Sum of a Triangular Number and a specific constant (product of the two indexing factors)" and the "Sum of Two Triangular Numbers" in Infinite Ways.*

$$\mathbf{T_\lambda} = T_{3\lambda n} - (T_{2\lambda-1} + T_{2\lambda})T_n - \lambda n$$

$$\mathbf{T_n} = \frac{T_{3\lambda n} - (T_\lambda + \lambda n)}{(T_{2\lambda-1} + T_{2\lambda})} \qquad [\lambda \geq 1, n \geq 1]$$

Proof. Trivial from (1.6) □

Corollary 1.6. *For* $\lambda \geq 1, n \geq 1$

$$(\mathbf{2\lambda})^{\mathbf{2}}(\mathbf{n}+\mathbf{1})^{\mathbf{2}} = T_{3\lambda n} + T_{(3\lambda(n+1))} - (2T_\lambda + \lambda(2n+1)). \quad (1.7)$$

Proof.

$$\overset{[\mathbf{1}]}{(\mathbf{3}\lambda - \mathbf{1})}\mathbf{T_n} = \frac{((3\lambda-1)n+1)((3\lambda-1)n+2)}{2}$$

$$= (2\lambda)^2 T_n + T_\lambda + \lambda n \quad (1.8)$$

$$= (T_{2\lambda-1} + T_{2\lambda})T_n + T_\lambda + \lambda n = T_{3\lambda n}$$

$$[\lambda \geq 1, n \geq 1]$$

$$(\mathbf{2}\lambda)^{\mathbf{2}}\mathbf{T_n} + \mathbf{T}_\lambda + \lambda\mathbf{n} = T_{3\lambda n} \quad (1.9)$$

Substituting $n = n+1$ in (1.9)

$$(\mathbf{2\lambda})^{\mathbf{2}}\mathbf{T_{n+1}} + \mathbf{T_\lambda} + \boldsymbol{\lambda}(\mathbf{n}+\mathbf{1}) = T_{(3\lambda(n+1))} \quad (1.10)$$

adding the two equations substituting $T_n + T_{n+1} = (n+1)^2$ [Nicomachus] and simplifying we have

$$(\mathbf{2\lambda})^{\mathbf{2}}(\mathbf{n}+\mathbf{1})^{\mathbf{2}} = T_{3\lambda n} + T_{(3\lambda(n+1))} - (2T_\lambda + \lambda(2n+1))$$

$$[\lambda \geq 1, n \geq 1] \quad (1.11)$$

□

2 Two General Theorems on Triangular Numbers

Theorem 2.1. *The square of every odd number can be expressed as the difference of two Triangular Numbers.*

Proof. We have

$$(\mathbf{2}\boldsymbol{\xi} + \mathbf{1})^{\mathbf{2}}\mathbf{T_n} + \mathbf{T}_{\boldsymbol{\xi}} = T_{(3\xi n+1)} \tag{2.1}$$

Substituting $n = 1$ in (2.1), we have

$$(\mathbf{2}\boldsymbol{\xi} + \mathbf{1})^{\mathbf{2}} = T_{(3\xi+1)} - T_{\xi} \quad [\xi \geq 0] \tag{2.2}$$

The standard result that "the square of any odd multiple of 3 is the difference of two Triangular Numbers [i.e., $9(2n + 1)^2 = T_{9n+4} - T_{3n+1}$]" [Ref. 6], clearly is a cumbersome expression of a special case when the "odd multiples of 3" are the odd numbers concerned! Again, looking for General Symmetry we have. □

Theorem 2.2. *The square of every λ^{th} even number can be expressed as the difference of the '$(3\lambda)^{th}$, Triangular Number' and 'the sum of the λ^{th} Triangular Number and λ'.*

Proof. We have,

$$(\mathbf{2}\boldsymbol{\lambda})^{\mathbf{2}}\mathbf{T_n} + \mathbf{T}_{\boldsymbol{\lambda}} + \boldsymbol{\lambda}\mathbf{n} = T_{3\lambda n} \tag{2.3}$$

Substituting $n = 1$ in (2.3), we have

$$(\mathbf{2}\boldsymbol{\lambda})^{\mathbf{2}} = T_{3\lambda} - (T_{\lambda} + \lambda) \quad [\lambda \geq 1] \tag{2.4}$$

3 Infinite Root Levels and Layer Levels of Triangular Numbers

In this section we define various Root-Levels and Layer-Levels of Triangular Numbers and investigate them.

In ${}_{L}^{[R]}T_n$, R may be called the Root Level and L the Layer Level which define a specific sub-class of Triangular Numbers.

The sequence of Layer-Levels for each Root-Level Facilitates the Proof for All Root-Levels.

$$ {}_{\mathbf{1}}^{[\mathbf{1}]}\mathbf{T}_{\mathbf{n+1}} = \frac{(n+1)(n+2)}{2} \qquad [n = 0, 1, 2, \text{---}, \infty] $$

$$ = T_1, T_2, T_3, T_4, T_5, \text{---}, T_{1+(n-1)}, \text{---}, \infty $$

$$ {}_{\mathbf{2}}^{[\mathbf{1}]}\mathbf{T}_{\mathbf{n+1}} = \frac{(2n+1)(2n+2)}{2} \qquad [n = 0, 1, 2, \text{---}, \infty] $$

$$ = T_1, T_3, T_5, T_7, T_9, \text{---}, T_{1+2(n-1)}, \text{---}, \infty $$

$$ {}_{\mathbf{3}}^{[\mathbf{1}]}\mathbf{T}_{\mathbf{n+1}} = \frac{(3n+1)(3n+2)}{2} \qquad [n = 0, 1, 2, \text{---}, \infty] $$

$$ = T_1, T_4, T_7, T_{10}, T_{13}, \text{---}, T_{1+3(n-1)}, \text{---}, \infty $$

$$ {}_{\mathbf{4}}^{[\mathbf{1}]}\mathbf{T}_{\mathbf{n+1}} = \frac{(4n+1)(4n+2)}{2} \qquad [n = 0, 1, 2, \text{---}, \infty] $$

$$ = T_1, T_5, T_9, T_{13}, T_{17}, \text{---}, T_{1+4(n-1)}, \text{---}, \infty $$

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$$ {}_{\phi}^{[\mathbf{1}]}\mathbf{T_{n+1}} = \frac{(\phi n+1)(\phi n+2)}{2} \qquad [n = 0, 1, 2, ---, \infty] $$

$$ = T_1, T_{1+\phi}, T_{1+2\phi}, T_{1+3\phi}, T_{1+4\phi}, ---, T_{1+\phi(n-1)}, ---, \infty $$

$$ {}_{\mathbf{1}}^{[\mathbf{2}]}\mathbf{T_{n+1}} = \frac{(2n+1)(2n+2)}{2} \qquad [n = 0, 1, 2, ---, \infty] $$

$$ = T_1, T_3, T_5, T_7, T_9, ---, T_{1+2(n-1)}, ---, \infty $$

$$ {}_{\mathbf{2}}^{[\mathbf{2}]}\mathbf{T_{n+1}} = \frac{(4n+1)(4n+2)}{2} \qquad [n = 0, 1, 2, ---, \infty] $$

$$ = T_1, T_5, T_9, T_{13}, T_{17}, ---, T_{1+4(n-1)}, ---, \infty $$

$$ {}_{\mathbf{3}}^{[\mathbf{2}]}\mathbf{T_{n+1}} = \frac{(6n+1)(6n+2)}{2} \qquad [n = 0, 1, 2, ---, \infty] $$

$$ = T_1, T_7, T_{13}, T_{19}, T_{25}, ---, T_{1+6(n-1)}, ---, \infty $$

$$ {}_{\mathbf{4}}^{[\mathbf{2}]}\mathbf{T_{n+1}} = \frac{(8n+1)(8n+2)}{2} \qquad [n = 0, 1, 2, ---, \infty] $$

$$ = T_1, T_9, T_{17}, T_{25}, T_{33}, ---, T_{1+8(n-1)}, ---, \infty $$

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$$ {}_{\phi}^{[\mathbf{2}]}\mathbf{T_{n+1}} = \frac{(2\phi n+1)(2\phi n+2)}{2} \qquad [n = 0, 1, 2, ---, \infty] $$

$$ = T_1, T_{1+2\phi}, T_{1+4\phi}, T_{1+6\phi}, T_{1+8\phi}, ---, T_{1+(2\phi n-1)}, ---, \infty $$

$$ {}_{\mathbf{1}}^{[\mathbf{3}]}\mathbf{T}_{\mathbf{n+1}} = \frac{(3n+1)(3n+2)}{2} \qquad [n = 0, 1, 2, \text{---}, \infty] $$

$$ = T_1, T_4, T_7, T_{10}, T_{13}, \text{---}, T_{1+3(n-1)}, \text{---}, \infty $$

$$ {}_{\mathbf{2}}^{[\mathbf{3}]}\mathbf{T}_{\mathbf{n+1}} = \frac{(6n+1)(6n+2)}{2} \qquad [n = 0, 1, 2, \text{---}, \infty] $$

$$ = T_1, T_7, T_{13}, T_{19}, T_{25}, \text{---}, T_{1+6(n-1)}, \text{---}, \infty $$

$$ {}_{\mathbf{3}}^{[\mathbf{3}]}\mathbf{T}_{\mathbf{n+1}} = \frac{(9n+1)(9n+2)}{2} \qquad [n = 0, 1, 2, \text{---}, \infty] $$

$$ = T_1, T_{10}, T_{19}, T_{28}, T_{37}, \text{---}, T_{1+9(n-1)}, \text{---}, \infty $$

$$ {}_{\mathbf{4}}^{[\mathbf{3}]}\mathbf{T}_{\mathbf{n+1}} = \frac{(12n+1)(12n+2)}{2} \qquad [n = 0, 1, 2, \text{---}, \infty] $$

$$ = T_1, T_{13}, T_{25}, T_{37}, T_{49}, \text{---}, T_{1+12(n-1)}, \text{---}, \infty $$

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$$ {}_{\phi}^{[\mathbf{3}]}\mathbf{T}_{\mathbf{n+1}} = \frac{(3\phi n+1)(3\phi n+2)}{2} \qquad [n = 0, 1, 2, \text{---}, \infty] $$

$$ = T_1, T_{1+3\phi}, T_{1+6\phi}, T_{1+9\phi}, T_{1+12\phi}, \text{---}, T_{1+3\phi(n-1)}, \text{---}, \infty $$

$$ {}_{\mathbf{1}}^{[\mathbf{4}]}\mathbf{T}_{\mathbf{n+1}} = \frac{(4n+1)(4n+2)}{2} \qquad [n = 0, 1, 2, \text{---}, \infty] $$

$$ = T_1, T_5, T_9, T_{13}, T_{17}, \text{---}, T_{1+4(n-1)}, \text{---}, \infty $$

$$\mathbf{2}^{[\mathbf{4}]}\mathbf{T_{n+1}} = \frac{(8n+1)(8n+2)}{2} \qquad [n = 0, 1, 2, ---, \infty]$$

$$= T_1, T_9, T_{17}, T_{25}, T_{33}, ---, T_{1+8(n-1)}, ---, \infty$$

$$\mathbf{3}^{[\mathbf{4}]}\mathbf{T_{n+1}} = \frac{(12n+1)(12n+2)}{2} \qquad [n = 0, 1, 2, ---, \infty]$$

$$= T_1, T_{13}, T_{25}, T_{37}, T_{49}, ---, T_{1+12(n-1)}, ---, \infty$$

$$\mathbf{4}^{[\mathbf{4}]}\mathbf{T_{n+1}} = \frac{(16n+1)(16n+2)}{2} \qquad [n = 0, 1, 2, ---, \infty]$$

$$= T_1, T_{17}, T_{33}, T_{49}, T_{65}, ---, T_{1+16(n-1)}, ---, \infty$$

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$$\phi^{[\mathbf{4}]}\mathbf{T_{n+1}} = \frac{(\phi 4n+1)(\phi 4n+2)}{2} \qquad [n = 0, 1, 2, ---, \infty]$$

$$= T_1, T_{1+4\phi}, T_{1+8\phi}, T_{1+12\phi}, T_{1+16\phi}, ---, T_{1+4\phi(n-1)}, ---, \infty$$

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$$\mathbf{1}^{[\mathbf{R}]}\mathbf{T_{n+1}} = \frac{(Rn+1)(Rn+2)}{2} \qquad [n = 0, 1, 2, ---, \infty]$$

$$= T_1, T_{1+R}, T_{1+2R}, T_{1+3R}, T_{1+4R}, ---, T_{1+R(n-1)}, ---, \infty$$

$$\mathbf{2}^{[\mathbf{R}]}\mathbf{T_{n+1}} = \frac{(2Rn+1)(2Rn+2)}{2} \qquad [n = 0, 1, 2, ---, \infty]$$

$$= T_1, T_{1+2R}, T_{1+4R}, T_{1+6R}, T_{1+8R}, ---, T_{1+2R(n-1)}, ---, \infty$$

$$\mathbf{{}_{3}^{[R]}T_{n+1}} = \frac{(3Rn+1)(3Rn+2)}{2} \qquad [n = 0, 1, 2, \text{---}, \infty]$$

$$= T_1, T_{1+3R}, T_{1+6R}, T_{1+9R}, T_{1+12R}, \text{---}, T_{1+3R(n-1)}, \text{---}, \infty$$

$$\mathbf{{}_{4}^{[R]}T_{n+1}} = \frac{(4Rn+1)(4Rn+2)}{2} \qquad [n = 0, 1, 2, \text{---}, \infty]$$

$$= T_1, T_{1+4R}, T_{1+8R}, T_{1+12R}, T_{1+16R}, \text{---}, T_{1+4R(n-1)}, \text{---}, \infty$$

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$$\mathbf{{}_{\phi}^{[R]}T_{n+1}} = \frac{(\phi Rn+1)(\phi Rn+2)}{2} \qquad [n = 0, 1, 2, \text{---}, \infty]$$

$$= T_1, T_{1+\phi R}, T_{1+2\phi R}, T_{1+3\phi R}, T_{1+4\phi R}, \text{---}, T_{1+\phi R(n-1)}, \text{---}, \infty$$

$$[\phi = 1, 2, \text{---}\infty]$$

$$[R = 1, 2, \text{---}, \infty]$$

4 The Infinite Triangular Number Classes in terms of Binomial Coefficients

We can easily prove by Repeated Induction the following results.

$$[n = 0, 1, 2, \text{---}, \infty]$$

$$\mathbf{{}_{1}^{[1]}T_{n+1}} = \binom{n+2}{2} \quad = T_{n+1} \quad \text{[This is the standard result. Ref.6)]}$$

$$\mathbf{{}_{2}^{[1]}T_{n+1}} = \binom{2n+2}{2}$$

$$\mathbf{{}_{3}^{[1]}T_{n+1}} = \binom{3n+2}{2}$$

$$\mathbf{{}_{4}^{[1]}T_{n+1}} = \binom{4n+2}{2}$$

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$$\mathbf{{}_{\phi}^{[1]}T_{n+1}} = \binom{\phi n+2}{2}$$

$$\mathbf{{}_{1}^{[2]}T_{n+1}} = \binom{2n+2}{2}$$

$$\mathbf{{}_{2}^{[2]}T_{n+1}} = \binom{4n+2}{2}$$

$$\mathbf{{}_{3}^{[2]}T_{n+1}} = \binom{6n+2}{2}$$

$$\mathbf{{}_{4}^{[2]}T_{n+1}} = \binom{8n+2}{2}$$

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$$\mathbf{{}_{\phi}^{[2]}T_{n+1}} = \binom{2\phi n+2}{2}$$

$$\mathbf{{}_{1}^{[3]}T_{n+1}} = \binom{3n+2}{2}$$

$$\mathbf{2}^{[\mathbf{3}]}\mathbf{T}_{\mathbf{n+1}} = \binom{6n+2}{2}$$

$$\mathbf{3}^{[\mathbf{3}]}\mathbf{T}_{\mathbf{n+1}} = \binom{9n+2}{2}$$

$$\mathbf{4}^{[\mathbf{3}]}\mathbf{T}_{\mathbf{n+1}} = \binom{12n+2}{2}$$

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$$\phi^{[\mathbf{3}]}\mathbf{T}_{\mathbf{n+1}} = \binom{3\phi n+2}{2}$$

$$\mathbf{1}^{[\mathbf{4}]}\mathbf{T}_{\mathbf{n+1}} = \binom{4n+2}{2}$$

$$\mathbf{2}^{[\mathbf{4}]}\mathbf{T}_{\mathbf{n+1}} = \binom{8n+2}{2}$$

$$\mathbf{3}^{[\mathbf{4}]}\mathbf{T}_{\mathbf{n+1}} = \binom{12n+2}{2}$$

$$\mathbf{4}^{[\mathbf{4}]}\mathbf{T}_{\mathbf{n+1}} = \binom{16n+2}{2}$$

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$$\phi^{[\mathbf{4}]}\mathbf{T}_{\mathbf{n+1}} = \binom{4\phi n+2}{2}$$

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The result if True for k ie.

$$ {}^{\phi}\overset{[\mathbf{k}]}{\mathbf{T}}_{\mathbf{n+1}} = \binom{k\phi n+2}{2} \Rightarrow \quad {}^{\phi}\overset{[k+1]}{T}_{n+1} $$

$$ = \binom{(k+1)\phi n+2}{2} $$

is also True for $k+1$ proving the General Result for $R = 1, 2,$ ---∞

$$ {}^{\mathbf{1}}\overset{[\mathbf{R}]}{\mathbf{T}}_{\mathbf{n+1}} = \binom{Rn+2}{2} $$

$$ {}^{\mathbf{2}}\overset{[\mathbf{R}]}{\mathbf{T}}_{\mathbf{n+1}} = \binom{2Rn+2}{2} $$

$$ {}^{\mathbf{3}}\overset{[\mathbf{R}]}{\mathbf{T}}_{\mathbf{n+1}} = \binom{3Rn+2}{2} $$

$$ {}^{\mathbf{4}}\overset{[\mathbf{R}]}{\mathbf{T}}_{\mathbf{n+1}} = \binom{4Rn+2}{2} $$

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$$ {}^{\phi}\overset{[\mathbf{R}]}{\mathbf{T}}_{\mathbf{n+1}} = \binom{\phi Rn+2}{2} \qquad [\phi = 1, 2, \text{---}, \infty] $$

$$ [R = 1, 2, \text{---}, \infty] $$

5 The Formulae for the Sum up to the n^{th} term of Infinite Root Levels and Layer Levels of Triangular Numbers

In this section we derive the Formulae for the Sum up to the n^{th} term of Infinite Root Levels and Layer Levels of Triangular Numbers. Aryabhatta's traditional formula for the Sum of Triangular Numbers is assumed.

For this purpose we need to consider a simple case of General Arithmetic cum Arithmetic Inductive Progressions. [Ref.7]

Let

$$[\mathbf{P}]_{\mathbf{n}} = a + b(n-1) + \frac{c(n-1)(n-2)}{2} \tag{5.1}$$

$[a, b$ and c are algebraic numbers.$]$

[Substituting $a = 1, b = 2$ and $c = 1$ we get the sequence of the set of All Triangular Numbers **T**.]

Now we can easily prove by Induction that the sum of this progression up to n terms is

$$\begin{aligned}\mathbf{S_n}[\mathbf{P}] = an + b\left[\frac{n(n-1)}{2}\right] \\ + \frac{c(n-1)(n-2)}{2} + \frac{2c(n-2)(n-3)}{2} \\ + \frac{3c(n-1)(n-2)}{2} \cdots \frac{(n-2)c(2)(1)}{2}.\end{aligned}$$

$$= an + b\left[\frac{n(n-1)}{2}\right] + c\sum_{1}^{n-2} i\frac{(n-i)(n-(i+1))}{2}$$

$$= an + b\left[\frac{n(n-1)}{2!}\right] + c\left[\frac{n(n-1)(n-2)}{3!}\right]$$

$$= \underset{\boxed{n \geq 1}}{\binom{a}{1}\binom{n}{1}} + \underset{\boxed{n \geq 2}}{\binom{b}{1}\binom{n}{2}} + \underset{\boxed{n \geq 3}}{\binom{c}{1}\binom{n}{3}} \qquad (5.2)$$

$$\sum_{\mathbf{0}}^{\mathbf{n-1}} {}^{\mathbf{1}}\mathbf{T}^{[\mathbf{1}]}_{\mathbf{i}+\mathbf{1}} = \sum_{0}^{n-1} \frac{(i+1)(i+2)}{2} \qquad [n = 1, 2, ---, \infty]$$

$$= \sum_{1}^{n} T_{1+(i-1)} = n + (T_2 - T_1)\left[\frac{n(n-1)}{2}\right] + 1^2\sum_{1}^{n-2} i\frac{(n-i)(n-(i+1))}{2}$$

Substituting $a = 1, b = 2, c = 1^2 = 1$, in (5.2) we have

$$= n + 2\left[\frac{n(n-1)}{2}\right] + \sum_{1}^{n-2} i\frac{(n-i)(n-(i+1))}{2}$$

$$= n + 2\left[\frac{n(n-1)}{2!}\right] + \left[\frac{n(n-1)(n-2)}{3!}\right]$$

$$= \underset{\boxed{n \geq 1}}{\binom{n}{1}} + \underset{\boxed{n \geq 2}}{\binom{2}{1}\binom{n}{2}} + \underset{\boxed{n \geq 3}}{\binom{n}{3}}$$

$$\sum_{\mathbf{0}}^{\mathbf{n-1}} {}^{\mathbf{2}}\mathbf{T}^{[\mathbf{1}]}_{\mathbf{i}\,+\,\mathbf{1}} = \sum_{0}^{n-1} \frac{(2i+1)(2i+2)}{2} \qquad [n = 1, 2, ---, \infty]$$

$$= \sum_{1}^{n} T_{1\,+\,2(i-1)}$$

$$= n + (T_3 - T_1)\left[\frac{n(n-1)}{2}\right] + 2^2 \sum_{1}^{n-2} i\frac{(n-i)(n-(i+1))}{2}$$

Substituting $a = 1$, $b = 5$, $c = 2^2 = 4$, in (5.2) we have

$$= n + 5\left[\frac{n(n-1)}{2}\right] + 4\sum_{1}^{n-2} i\frac{(n-i)(n-(i+1))}{2}$$

$$= n + 5\left[\frac{n(n-1)}{2!}\right] + 4\left[\frac{n(n-1)(n-2)}{3!}\right]$$

$$= \underset{\boxed{n \geq 1}}{\binom{n}{1}} + \underset{\boxed{n \geq 2}}{\binom{5}{1}\binom{n}{2}} + \underset{\boxed{n \geq 3}}{\binom{4}{1}\binom{n}{3}}$$

$$\sum_{\mathbf{0}}^{\mathbf{n-1}} {}^{\mathbf{3}}\mathbf{T}^{[\mathbf{1}]}_{\mathbf{i}\,+\,\mathbf{1}} = \sum_{0}^{n-1} \frac{(3i+1)(3i+2)}{2} \qquad [n = 1, 2, ---, \infty]$$

$$= \sum_{1}^{n} T_{1\,+\,3(i-1)} = n + (T_4 - T_1)\left[\frac{n(n-1)}{2}\right] + 3^2 \sum_{1}^{n-2} i\frac{(n-i)(n-(i+1))}{2}$$

Substituting $a = 1$, $b = 9$, $c = 3^2 = 9$, in (5.2) we have

$$= n + 9\left[\frac{n(n-1)}{2}\right] + 9\sum_{1}^{n-2} i\frac{(n-i)(n-(i+1))}{2}$$

$$= n + 9\left[\frac{n(n-1)}{2!}\right] + 9\left[\frac{n(n-1)(n-2)}{3!}\right]$$

$$= \underset{\boxed{n \geq 1}}{\binom{n}{1}} + \underset{\boxed{n \geq 2}}{\binom{9}{1}\binom{n}{2}} + \underset{\boxed{n \geq 3}}{\binom{4}{1}\binom{n}{3}}$$

$$\sum_{\mathbf{0}}^{\mathbf{n-1}} {}^{[\mathbf{1}]}_{\mathbf{4}}\mathbf{T}_{\mathbf{i}+\mathbf{1}} = \sum_{0}^{n-1} \frac{(4i+1)(4i+2)}{2} \qquad [n = 1, 2, ---, \infty]$$

$$= \sum_{1}^{n} T_{1+4(i-1)} = n + (T_5 - T_1)\left[\frac{n(n-1)}{2}\right] + 4^2\sum_{1}^{n-2} i\frac{(n-i)(n-(i+1))}{2}$$

Substituting $a = 1$, $b = 14$, $c = 4^2 = 16$, in (5.2) we have

$$= n + 14\left[\frac{n(n-1)}{2}\right] + 16\sum_{1}^{n-2} i\frac{(n-i)(n-(i+1))}{2}$$

$$= n + 14\left[\frac{n(n-1)}{2!}\right] + 16\left[\frac{n(n-1)(n-2)}{3!}\right]$$

$$= \underset{\boxed{n \geq 1}}{\binom{n}{1}} + \underset{\boxed{n \geq 2}}{\binom{14}{1}\binom{n}{2}} + \underset{\boxed{n \geq 3}}{\binom{16}{1}\binom{n}{3}}$$

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$$\sum_{\mathbf{0}}^{\mathbf{n-1}} {}^{\phi}\mathbf{T}^{[\mathbf{1}]}_{\mathbf{i+1}} = \sum_{0}^{n-1} \frac{(\phi i+1)(\phi i+2)}{2} \qquad [n = 1, 2, ---, \infty]$$

$$= \sum_{1}^{n} T_{1+\phi(i-1)} = n + (T_{1+\phi} - T_1)\left[\frac{n(n-1)}{2}\right] + \phi^2 \sum_{1}^{n-2} i \frac{(n-i)(n-(i+1))}{2}$$

Substituting $a = 1$, $b = \left[\frac{(\phi+1)(\phi+2)}{2} - 1\right]$, $c = \phi^2$, in (5.2) we have

$$= n\left[\frac{(\phi+1)(\phi+2)}{2} - 1\right]\left[\frac{n(n-1)}{2}\right] + \phi^2 \sum_{1}^{n-2} i \frac{(n-i)(n-(i+1))}{2}$$

$$= n + \left[\frac{(\phi+1)(\phi+2)}{2} - 1\right]\left[\frac{n(n-1)}{2!}\right] + \phi^2\left[\frac{n(n-1)(n-2)}{3!}\right]$$

$$= \underset{\boxed{n \geq 1}}{\binom{n}{1}} + \left(\begin{matrix}\left[\frac{(\phi+1)(\phi+2)}{2} - 1\right] \\ 1\end{matrix}\right) \underset{\boxed{n \geq 2}}{\binom{n}{2}} + \binom{\phi^2}{1} \underset{\boxed{n \geq 3}}{\binom{n}{3}}$$

$$\sum_{\mathbf{0}}^{\mathbf{n-1}} {}^{\mathbf{1}}\mathbf{T}^{[\mathbf{2}]}_{\mathbf{i+1}} = \sum_{0}^{n-1} \frac{(2i+1)(2i+2)}{2} \qquad [n = 1, 2, ---, \infty]$$

$$= \sum_{1}^{n} T_{1+2(i-1)} = n + (T_3 - T_1)\left[\frac{n(n-1)}{2}\right]$$

$$+2^2 \sum_{1}^{n-2} i \frac{(n-i)(n-(i+1))}{2}$$

Substituting $a = 1$, $b = 5$, $c = 2^2 = 4$, in (5.2) we have

$$= n + 5\left[\frac{n(n-1)}{2}\right] + 4\sum_{1}^{n-2} i \frac{(n-i)(n-(i+1))}{2}$$

$$= n + 5\left[\frac{n(n-1)}{2!}\right] + 4\left[\frac{n(n-1)(n-2)}{3!}\right]$$

$$= \underset{\boxed{n \geq 1}}{\binom{n}{1}} + \underset{\boxed{n \geq 2}}{\binom{5}{1}\binom{n}{2}} + \underset{\boxed{n \geq 3}}{\binom{4}{1}\binom{n}{3}}$$

$$\sum_{\mathbf{0}}^{\mathbf{n-1}} {}^{\mathbf{2}}\mathbf{T}^{[\mathbf{2}]}_{\mathbf{i}+\mathbf{1}} = \sum_{0}^{n-1} \frac{(4i+1)(4i+2)}{2} \qquad [n = 1, 2, ---, \infty]$$

$$= \sum_{1}^{n} T_{1+4(i-1)} = n + (T_5 - T_1)\left[\frac{n(n-1)}{2}\right]$$

$$+4^2 \sum_{1}^{n-2} i \frac{(n-i)(n-(i+1))}{2}$$

Substituting $a = 1$, $b = 14$, $c = 4^2 = 16$, in (5.2) we have

$$= n + 14\left[\frac{n(n-1)}{2}\right] + 16\sum_{1}^{n-2} i \frac{(n-i)(n-(i+1))}{2}$$

$$= n + 14\left[\frac{n(n-1)}{2!}\right] + 16\left[\frac{n(n-1)(n-2)}{3!}\right]$$

$$= \binom{n}{1} + \binom{14}{1}\binom{n}{2} + \binom{16}{1}\binom{n}{3}$$

$$\boxed{n \geq 1} \quad \boxed{n \geq 2} \quad \boxed{n \geq 3}$$

$$\sum_{\mathbf{0}}^{\mathbf{n-1}} {}^{\mathbf{3}}\overset{[\mathbf{2}]}{\mathbf{T}}_{\mathbf{i}+\mathbf{1}} = \sum_{0}^{n-1} \frac{(6i+1)(6i+2)}{2} \qquad [n = 1, 2, \text{---}, \infty]$$

$$= \sum_{1}^{n} T_{1+6(i-1)} = n + (T_7 - T_1)\left[\frac{n(n-1)}{2}\right]$$

$$+6^2 \sum_{1}^{n-2} i\frac{(n-i)(n-(i+1))}{2}$$

Substituting $a = 1$, $b = 27$, $c = 6^2 = 36$, in (5.2) we have

$$= n + 27\left[\frac{n(n-1)}{2}\right] + 36\sum_{1}^{n-2} i\frac{(n-i)(n-(i+1))}{2}$$

$$= n + 27\left[\frac{n(n-1)}{2!}\right] + 36\left[\frac{n(n-1)(n-2)}{3!}\right]$$

$$= \binom{n}{1} + \binom{27}{1}\binom{n}{2} + \binom{36}{1}\binom{n}{3}$$

$$\boxed{n \geq 1} \quad \boxed{n \geq 2} \quad \boxed{n \geq 3}$$

$$\sum_{\mathbf{0}}^{\mathbf{n-1}} {}^{\mathbf{4}}\overset{[\mathbf{2}]}{\mathbf{T}}_{\mathbf{i}+\mathbf{1}} = \sum_{0}^{n-1} \frac{(8i+1)(8i+2)}{2} \qquad [n = 1, 2, \text{---}, \infty]$$

$$= \sum_{1}^{n} T_{1+8(i-1)} = n + (T_9 - T_1)\left[\frac{n(n-1)}{2}\right]$$

$$+8^2 \sum_{1}^{n-2} i\frac{(n-i)(n-(i+1))}{2}$$

Substituting $a = 1$, $b = 44$, $c = 8^2 = 64$, in (5.2) we have

$$= n + 44\left[\frac{n(n-1)}{2}\right] + 64\sum_{1}^{n-2} i\frac{(n-i)(n-(i+1))}{2}$$

$$= n + 44\left[\frac{n(n-1)}{2!}\right] + 64\left[\frac{n(n-1)(n-2)}{3!}\right]$$

$$= \underset{\boxed{n \geq 1}}{\binom{n}{1}} + \underset{\boxed{n \geq 2}}{\binom{44}{1}\binom{n}{2}} + \underset{\boxed{n \geq 3}}{\binom{64}{1}\binom{n}{3}}$$

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$$\sum_{\mathbf{0}}^{\mathbf{n-1}} {}^{\phi}\mathbf{T}^{[\mathbf{2}]}_{\mathbf{i+1}} = \sum_{0}^{n-1} \frac{(2\phi i+1)(2\phi i+2)}{2} \qquad [n = 1, 2, ---, \infty]$$

$$= \sum_{1}^{n} T_{1+2\phi(i-1)} = n + (T_{1+2\phi} - T_1)\left[\frac{n(n-1)}{2}\right] + (2\phi)^2\sum_{1}^{n-2} i\frac{(n-i)(n-(i+1))}{2}$$

Substituting $a = 1$, $b = \left[\frac{(2\phi+1)(2\phi+2)}{2} - 1\right]$, $c = (2\phi)^2$, in (5.2) we have

$$= n + \left[\frac{(2\phi+1)(2\phi+2)}{2} - 1\right]\left[\frac{n(n-1)}{2}\right] + (2\phi)^2\sum_{1}^{n-2} i\frac{(n-i)(n-(i+1))}{2}$$

$$= n + \left[\frac{(2\phi+1)(2\phi+2)}{2} - 1\right]\left[\frac{n(n-1)}{2!}\right]$$

$$+(2\phi)^2\left[\frac{n(n-1)(n-2)}{3!}\right]$$

$$= \underset{\boxed{n\geq 1}}{\binom{n}{1}} + \left(\begin{matrix}\left[\frac{(2\phi+1)(2\phi+2)}{2} - 1\right] \\ 1\end{matrix}\right)\underset{\boxed{n\geq 2}}{\binom{n}{2}} + \binom{(2\phi)^2}{1}\underset{\boxed{n\geq 3}}{\binom{n}{3}}$$

$$\sum_{\mathbf{0}}^{\mathbf{n-1}} {}^{\mathbf{1}}\overset{[\mathbf{3}]}{\mathbf{T}}_{\mathbf{i}\,+\,\mathbf{1_2}} = \sum_{0}^{n-1}\frac{(3i+1)(3i+2)}{2} \qquad [n = 1, 2, \text{---}, \infty]$$

$$= \sum_{1}^{n} T_{1\,+\,3(i-1)} = n + (T_4 - T_1)\left[\frac{n(n-1)}{2}\right]$$

$$+3^2\sum_{1}^{n-2} i\frac{(n-i)(n-(i+1))}{2}$$

Substituting $a = 1$, $b = 9$, $c = 3^2 = 9$, in (5.2) we have

$$= n + 9\left[\frac{n(n-1)}{2}\right] + 9\sum_{1}^{n-2} i\frac{(n-i)(n-(i+1))}{2}$$

$$= n + 9\left[\frac{n(n-1)}{2!}\right] + 9\left[\frac{n(n-1)(n-2)}{3!}\right]$$

$$= \underset{\boxed{n\geq 1}}{\binom{n}{1}} + \binom{9}{1}\underset{\boxed{n\geq 2}}{\binom{n}{2}} + \binom{9}{1}\underset{\boxed{n\geq 3}}{\binom{n}{3}}$$

$$\sum_{\mathbf{0}}^{\mathbf{n-1}} {}^{\mathbf{2}}\overset{[\mathbf{3}]}{\mathbf{T}}_{\mathbf{i}\,+\,\mathbf{1}} = \sum_{0}^{n-1}\frac{(6i+1)(6i+2)}{2} \qquad [n = 1, 2, \text{---}, \infty]$$

$$= \sum_{1}^{n} T_{1+6(i-1)} = n + (T_7 - T_1)\left[\frac{n(n-1)}{2}\right]$$
$$+6^2 \sum_{1}^{n-2} i\frac{(n-i)(n-(i+1))}{2}$$

Substituting $a = 1$, $b = 27$, $c = 6^2 = 36$, in (5.2) we have

$$= n + 27\left[\frac{n(n-1)}{2}\right] + 36\sum_{1}^{n-2} i\frac{(n-i)(n-(i+1))}{2}$$

$$= n + 27\left[\frac{n(n-1)}{2!}\right] + 36\left[\frac{n(n-1)(n-2)}{3!}\right]$$

$$= \underset{\boxed{n\geq 1}}{\binom{n}{1}} + \underset{\boxed{n\geq 2}}{\binom{27}{1}\binom{n}{2}} + \underset{\boxed{n\geq 3}}{\binom{36}{1}\binom{n}{3}}$$

$$\sum_{\mathbf{0}}^{\mathbf{n-1}} {}^{\mathbf{3}}\mathbf{T}^{[\mathbf{3}]}_{\mathbf{i+1}} = \sum_{0}^{n-1} \frac{(9i+1)(9i+2)}{2} \qquad [n = 1, 2, ---, \infty]$$

$$= \sum_{1}^{n} T_{1+9(i-1)} = n + (T_{10} - T_1)\left[\frac{n(n-1)}{2}\right]$$
$$+9^2 \sum_{1}^{n-2} i\frac{(n-i)(n-(i+1))}{2}$$

Substituting $a = 1$, $b = 54$, $c = 9^2 = 81$, in (5.2) we have

$$= n + 54\left[\frac{n(n-1)}{2}\right] + 81\sum_{1}^{n-2} i\frac{(n-i)(n-(i+1))}{2}$$

$$= n + 54\left[\frac{n(n-1)}{2!}\right] + 81\left[\frac{n(n-1)(n-2)}{3!}\right]$$

$$= \underset{\boxed{n \geq 1}}{\binom{n}{1}} + \underset{\boxed{n \geq 2}}{\binom{54}{1}\binom{n}{2}} + \underset{\boxed{n \geq 3}}{\binom{81}{1}\binom{n}{3}}$$

$$\sum_{\mathbf{0}}^{\mathbf{n-1}} {}^{\mathbf{4}}\mathbf{T}^{[\mathbf{3}]}_{\mathbf{i}+\mathbf{1}} = \sum_{0}^{n-1} \frac{(12i+1)(12i+2)}{2} \qquad [n = 1, 2, \text{---}, \infty]$$

$$= \sum_{1}^{n} T_{1+12(i-1)} = n + (T_{13} - T_1)\left[\frac{n(n-1)}{2}\right] + (12)^2 \sum_{1}^{n-2} i \frac{(n-i)(n-(i+1))}{2}$$

Substituting $a = 1$, $b = 90$, $c = (12)^2 = 144$, in (5.2) we have

$$= n + 90\left[\frac{n(n-1)}{2}\right] + 144 \sum_{1}^{n-2} i \frac{(n-i)(n-(i+1))}{2}$$

$$= n + 90\left[\frac{n(n-1)}{2!}\right] + 144\left[\frac{n(n-1)(n-2)}{3!}\right]$$

$$= \underset{\boxed{n \geq 1}}{\binom{n}{1}} + \underset{\boxed{n \geq 2}}{\binom{90}{1}\binom{n}{2}} + \underset{\boxed{n \geq 3}}{\binom{144}{1}\binom{n}{3}}$$

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$$\sum_{\mathbf{0}}^{\mathbf{n-1}} {}^{\phi}\mathbf{T}^{[\mathbf{3}]}_{\mathbf{i}+\mathbf{1}} = \sum_{0}^{n-1} \frac{(3\phi i+1)(3\phi i+2)}{2} \qquad [n = 1, 2, \text{---}, \infty]$$

$$= \sum_{1}^{n} T_{1+3\phi(i-1)} = n + (T_{1+3\phi} - T_1)\left[\frac{n(n-1)}{2}\right]$$

$$+(3\phi)^2 \sum_{1}^{n-2} i\frac{(n-i)(n-(i+1))}{2}$$

Substituting $a = 1$, $b = \left[\frac{(3\phi+1)(3\phi+2)}{2} - 1\right]$, $c = (3\phi)^2$, in (5.2) we have

$$= n + \left[\frac{(3\phi+1)(3\phi+2)}{2} - 1\right]\left[\frac{n(n-1)}{2}\right]$$

$$+(3\phi)^2 \sum_{1}^{n-2} i\frac{(n-i)(n-(i+1))}{2}$$

$$= n + \left[\frac{(3\phi+1)(3\phi+2)}{2} - 1\right]\left[\frac{n(n-1)}{2!}\right]$$

$$+(3\phi)^2 \left[\frac{n(n-1)(n-2)}{3!}\right]$$

$$= \underset{\boxed{n \geq 1}}{\binom{n}{1}} + \left(\begin{matrix}\left[\frac{(3\phi+1)(3\phi+2)}{2} - 1\right] \\ 1\end{matrix}\right) \underset{\boxed{n \geq 2}}{\binom{n}{2}} + \underset{\boxed{n \geq 3}}{\binom{(3\phi)^2}{1}\binom{n}{3}}$$

$$\sum_{\mathbf{0}}^{\mathbf{n-1}} {}_{\mathbf{1}}^{[\mathbf{4}]}\mathbf{T}_{\mathbf{i}\,+\,\mathbf{1}} = \sum_{0}^{n-1} \frac{(4i+1)(4i+2)}{2} \qquad [n = 1, 2, \text{- - -}, \infty]$$

$$= \sum_{1}^{n} T_{1+4(i-1)} = n + (T_5 - T_1)\left[\frac{n(n-1)}{2}\right]$$

$$+4^2 \sum_{1}^{n-2} i\frac{(n-i)(n-(i+1))}{2}$$

Substituting $a = 1$, $b = 14$, $c = 4^2 = 16$, in (5.2) we have

$$= n + 14\left[\frac{n(n-1)}{2}\right] + 16\sum_{1}^{n-2} i\frac{(n-i)(n-(i+1))}{2}$$

$$= n + 14\left[\frac{n(n-1)}{2!}\right] + 16\left[\frac{n(n-1)(n-2)}{3!}\right]$$

$$= \underset{\boxed{n\geq 1}}{\binom{n}{1}} + \underset{\boxed{n\geq 2}}{\binom{14}{1}\binom{n}{2}} + \underset{\boxed{n\geq 3}}{\binom{16}{1}\binom{n}{3}}$$

$$\sum_{\mathbf{0}}^{\mathbf{n-1}} {}^{\mathbf{2}}\mathbf{T}^{[\mathbf{4}]}_{\mathbf{i}+\mathbf{1}} = \sum_{0}^{n-1}\frac{(8i+1)(8i+2)}{2} \qquad [n = 1, 2, ---, \infty]$$

$$= \sum_{1}^{n} T_{1+8(i-1)} = n + (T_9 - T_1)\left[\frac{n(n-1)}{2}\right] + 8^2\sum_{1}^{n-2} i\frac{(n-i)(n-(i+1))}{2}$$

Substituting $a = 1$, $b = 44$, $c = (8)^2 = 64$, in (5.2) we have

$$= n + 44\left[\frac{n(n-1)}{2}\right] + 64\sum_{1}^{n-2} i\frac{(n-i)(n-(i+1))}{2}$$

$$= n + 44\left[\frac{n(n-1)}{2!}\right] + 64\left[\frac{n(n-1)(n-2)}{3!}\right]$$

$$= \underset{\boxed{n\geq 1}}{\binom{n}{1}} + \underset{\boxed{n\geq 2}}{\binom{44}{1}\binom{n}{2}} + \underset{\boxed{n\geq 3}}{\binom{64}{1}\binom{n}{3}}$$

$$\sum_{\mathbf{0}}^{\mathbf{n-1}} {}^{\mathbf{3}}\mathbf{T}^{[\mathbf{4}]}_{\mathbf{i}+\mathbf{1}} = \sum_{0}^{n-1}\frac{(12i+1)(12i+2)}{2} \qquad [n = 1, 2, ---, \infty]$$

$$= \sum_{1}^{n} T_{1+12(i-1)}$$

$$= n + (T_{13} - T_1)\left[\frac{n(n-1)}{2}\right]$$

$$+(12)^2 \sum_{1}^{n-2} i\frac{(n-i)(n-(i+1))}{2}$$

Substituting $a = 1$, $b = 90$, $c = (12)^2 = 144$, in (5.2) we have

$$= n + 90\left[\frac{n(n-1)}{2}\right] + 144\sum_{1}^{n-2} i\frac{(n-i)(n-(i+1))}{2}$$

$$= n + 90\left[\frac{n(n-1)}{2!}\right] + 144\left[\frac{n(n-1)(n-2)}{3!}\right]$$

$$= \underset{\boxed{n \geq 1}}{\binom{n}{1}} + \underset{\boxed{n \geq 2}}{\binom{90}{1}\binom{n}{2}} + \underset{\boxed{n \geq 3}}{\binom{144}{1}\binom{n}{3}}$$

$$\sum_{\mathbf{0}}^{\mathbf{n-1}} {}^{\mathbf{4}}\mathbf{T}^{[\mathbf{4}]}_{\mathbf{i}+\mathbf{1}} = \sum_{0}^{n-1} \frac{(16i+1)(16i+2)}{2} \qquad [n = 1, 2, ---, \infty]$$

$$= \sum_{1}^{n} T_{1+16(i-1)} = n + (T_{17} - T_1)\left[\frac{n(n-1)}{2}\right]$$

$$+(16)^2 \sum_{1}^{n-2} i\frac{(n-i)(n-(i+1))}{2}$$

Substituting $a = 1$, $b = 152$, $c = (16)^2 = 256$, in (5.2) we have

$$= n + 152\left[\frac{n(n-1)}{2}\right] + 256\sum_{1}^{n-2} i\frac{(n-i)(n-(i+1))}{2}$$

$$= n + 152\left[\frac{n(n-1)}{2!}\right] + 256\left[\frac{n(n-1)(n-2)}{3!}\right]$$

$$= \underset{\boxed{n\geq 1}}{\binom{n}{1}} + \underset{\boxed{n\geq 2}}{\binom{152}{1}\binom{n}{2}} + \underset{\boxed{n\geq 3}}{\binom{256}{1}\binom{n}{3}}$$

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$$\sum_{\mathbf{0}}^{\mathbf{n-1}} {}^{\phi}\mathbf{T}^{[\mathbf{4}]}_{\mathbf{i}+\mathbf{1}} = \sum_{0}^{n-1} \frac{(\phi 4i+1)(\phi 4i+2)}{2} \qquad [n = 1, 2, ---, \infty]$$

$$= \sum_{1}^{n} T_{1+4\phi(i-1)} = n + (T_{1+4\phi} - T_1)\left[\frac{n(n-1)}{2}\right] + (4\phi)^2 \sum_{1}^{n-2} i \frac{(n-i)(n-(i+1))}{2}$$

Substituting $a = 1$, $b = \left[\frac{(4\phi+1)(4\phi+2)}{2} - 1\right]$, $c = (4\phi)^2$, in (5.2) we have

$$= n + \left[\frac{(4\phi+1)(4\phi+2)}{2} - 1\right]\left[\frac{n(n-1)}{2}\right] + (4\phi)^2 \sum_{1}^{n-2} i \frac{(n-i)(n-(i+1))}{2}$$

$$= n + \left[\frac{(4\phi+1)(4\phi+2)}{2} - 1\right]\left[\frac{n(n-1)}{2!}\right] + (4\phi)^2\left[\frac{n(n-1)(n-2)}{3!}\right]$$

$$= \underset{\boxed{n \geq 1}}{\binom{n}{1}} + \left(\begin{matrix}\left[\dfrac{(4\phi+1)(4\phi+2)}{2} - 1\right] \\ 1\end{matrix}\right) \underset{\boxed{n \geq 2}}{\binom{n}{2}} + \underset{\boxed{n \geq 3}}{\binom{(4\phi)^2}{1}\binom{n}{3}}$$

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$$\sum_{\mathbf{0}}^{\mathbf{n-1}} {}^{[\mathbf{R}]}_{\mathbf{1}}\mathbf{T}_{\mathbf{i}\,+\,\mathbf{1}} = \sum_{0}^{n-1} \frac{(Ri+1)(Ri+2)}{2} \qquad [n = 1, 2, \text{- - -}, \infty]$$

$$= \sum_{1}^{n} T_{1+R(i-1)}$$

$$= n + (T_{1+R} - T_1)\left[\frac{n(n-1)}{2}\right] + R^2 \sum_{1}^{n-2} i \frac{(n-i)(n-(i+1))}{2}$$

Substituting $a = 1$, $b = \left[\dfrac{(R+1)(R+2)}{2} - 1\right]$, $c = R^2$, in (5.2) we have

$$= n + \left[\frac{(R+1)(R+2)}{2} - 1\right]\left[\frac{n(n-1)}{2}\right] + R^2 \sum_{1}^{n-2} i \frac{(n-i)(n-(i+1))}{2}$$

$$= n + \left[\frac{(R+1)(R+2)}{2} - 1\right]\left[\frac{n(n-1)}{2!}\right] + R^2 \left[\frac{n(n-1)(n-2)}{3!}\right]$$

$$= \underset{\boxed{n \geq 1}}{\binom{n}{1}} + \left(\begin{matrix}\left[\dfrac{(R+1)(R+2)}{2} - 1\right] \\ 1\end{matrix}\right) \underset{\boxed{n \geq 2}}{\binom{n}{2}} + \underset{\boxed{n \geq 3}}{\binom{R^2}{1}\binom{n}{3}}$$

$$\sum_{\mathbf{0}}^{\mathbf{n-1}} {}^{\mathbf{2}}\mathbf{T}^{[\mathbf{R}]}_{\mathbf{i}+\mathbf{1}} = \sum_{0}^{n-1} \frac{(2Ri+1)(2Ri+2)}{2} \qquad [n = 1, 2, \text{---}, \infty]$$

$$= \sum_{1}^{n} T_{1+2R(i-1)} = n + (T_{1+2R} - T_1)\left[\frac{n(n-1)}{2}\right] + (2R)^2 \sum_{1}^{n-2} i\frac{(n-i)(n-(i+1))}{2}$$

Substituting $a = 1$, $b = \left[\frac{(2R+1)(2R+2)}{2} - 1\right]$, $c = (2R)^2$, in (5.2) we have

$$= n + \left[\frac{(2R+1)(2R+2)}{2} - 1\right]\left[\frac{n(n-1)}{2}\right] + (2R)^2 \sum_{1}^{n-2} i\frac{(n-i)(n-(i+1))}{2}$$

$$= n + \left[\frac{(2R+1)(2R+2)}{2} - 1\right]\left[\frac{n(n-1)}{2!}\right] + (2R)^2\left[\frac{n(n-1)(n-2)}{3!}\right]$$

$$= \underset{\boxed{n \geq 1}}{\binom{n}{1}} + \left(\begin{matrix}\left[\frac{(2R+1)(2R+2)}{2} - 1\right] \\ 1\end{matrix}\right)\underset{\boxed{n \geq 2}}{\binom{n}{2}} + \binom{(2R)^2}{1}\underset{\boxed{n \geq 3}}{\binom{n}{3}}$$

$$\sum_{\mathbf{0}}^{\mathbf{n-1}} {}^{\mathbf{3}}\mathbf{T}^{[\mathbf{R}]}_{\mathbf{i}+\mathbf{1}} = \sum_{0}^{n-1} \frac{(3Ri+1)(3Ri+2)}{2} \qquad [n = 1, 2, \text{---}, \infty]$$

$$= \sum_{1}^{n} T_{1+3R(i-1)} = n + (T_{1+3R} - T_1)\left[\frac{n(n-1)}{2}\right]$$

$$+(3R)^2 \sum_{1}^{n-2} i\frac{(n-i)(n-(i+1))}{2}$$

Substituting $a = 1$, $b = \left[\frac{(3R+1)(3R+2)}{2} - 1\right]$, $c = (3R)^2$, in (5.2) we have

$$= n + \left[\frac{(3R+1)(3R+2)}{2} - 1\right]\left[\frac{n(n-1)}{2}\right] + (3R)^2 \sum_{1}^{n-2} i\frac{(n-i)(n-(i+1))}{2}$$

$$= n + \left[\frac{(3R+1)(3R+2)}{2} - 1\right]\left[\frac{n(n-1)}{2!}\right] + (3R)^2\left[\frac{n(n-1)(n-2)}{3!}\right]$$

$$= \underset{\boxed{n \geq 1}}{\binom{n}{1}} + \underset{\boxed{n \geq 2}}{\left(\left[\frac{(3R+1)(3R+2)}{2} - 1\right] \atop 1\right)\binom{n}{2}} + \underset{\boxed{n \geq 3}}{\binom{(3R)^2}{1}\binom{n}{3}}$$

$$\sum_{\mathbf{0}}^{\mathbf{n-1}} {}^{\mathbf{4}}\mathbf{T}^{[\mathbf{R}]}_{\mathbf{i}+\mathbf{1}} = \sum_{0}^{n-1} \frac{(4Ri+1)(4Ri+2)}{2} \qquad [n = 1, 2, ---, \infty]$$

$$= \sum_{1}^{n} T_{1+4R(i-1)} = n + (T_{1+4R} - T_1)\left[\frac{n(n-1)}{2}\right] + (4R)^2 \sum_{1}^{n-2} i\frac{(n-i)(n-(i+1))}{2}$$

Substituting $a = 1$, $b = \left[\frac{(4R+1)(4R+2)}{2} - 1\right]$, $c = (4R)^2$, in (5.2) we have

$$= n + \left[\frac{(4R+1)(4R+2)}{2} - 1\right]\left[\frac{n(n-1)}{2}\right] + (4R)^2 \sum_{1}^{n-2} i\frac{(n-i)(n-(i+1))}{2}$$

$$= n + \left[\frac{(4R+1)(4R+2)}{2} - 1\right]\left[\frac{n(n-1)}{2!}\right] + (4R)^2\left[\frac{n(n-1)(n-2)}{3!}\right]$$

$$= \underset{\boxed{n \geq 1}}{\binom{n}{1}} + \left(\begin{matrix}\left[\frac{(4R+1)(4R+2)}{2} - 1\right] \\ 1\end{matrix}\right)\underset{\boxed{n \geq 2}}{\binom{n}{2}} + \binom{(4R)^2}{1}\underset{\boxed{n \geq 3}}{\binom{n}{3}}$$

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$$\sum_{\mathbf{0}}^{\mathbf{n-1}} {}^{\phi}\overset{[\mathbf{R}]}{\mathbf{T}}_{\mathbf{i+1}} = \sum_{0}^{n-1}\frac{(\phi R i+1)(\phi R i+2)}{2} \qquad [n = 1, 2, \text{---}, \infty]$$

$$= \sum_{1}^{n} T_{1+\phi R(i-1)} = n + (T_{1+\phi R} - T_1)\left[\frac{n(n-1)}{2}\right] + (\phi R)^2 \sum_{1}^{n-2} i\frac{(n-i)(n-(i+1))}{2}$$

Substituting $a = 1$, $b = \left[\frac{(\phi R+1)(\phi R+2)}{2} - 1\right]$, $c = (\phi R)^2$, in (5.2) we have

$$= n + \left[\frac{(\phi R+1)(\phi R+2)}{2} - 1\right]\left[\frac{n(n-1)}{2}\right] + (\phi R)^2 \sum_{1}^{n-2} i\frac{(n-i)(n-(i+1))}{2}$$

$$= n + \left[\frac{(\phi R+1)(\phi R+2)}{2} - 1\right]\left[\frac{n(n-1)}{2!}\right] + (\phi R)^2\left[\frac{n(n-1)(n-2)}{3!}\right]$$

$$= \underset{\boxed{n \geq 1}}{\binom{n}{1}} + \binom{\left[\frac{(\phi R+1)(\phi R+2)}{2} - 1\right]}{1}\underset{\boxed{n \geq 2}}{\binom{n}{2}} + \binom{(\phi R)^2}{1}\underset{\boxed{n \geq 3}}{\binom{n}{3}}$$

6 Some Interesting Properties of Triangular Numbers

We have proved Theorem 1.1 that
If $T = \{T_0, T_1, T_2.T_3, ---, T_n, ---, \infty\}[T_0 = 0]$ are Triangular Numbers then $(2\xi + 1)^2 T_n + T_\xi[\xi \geq 0, n \geq 0]$ are also Triangular Numbers.

We can utilize this theorem to generate Infinite Inductive Theorems as follows. The self-evident (though a bit cumbersome) inductive steps of the Proofs are omitted.

Theorem 6.1. *There Exist Infinite Triangular Numbers That Are the sum of three Triangular Numbers.*

Proof. Substituting $n = 1$ in "$(T_{2\xi} + T_{2\xi+1})T_n + T_\xi = T_{(3\xi n+1)}$"[1.1]

we have $T_{2\xi} + T_{2\xi+1} + T_\xi = T_{(3\xi+1)} \cdot [\xi \geq 1]$ proving the theorem. □

Theorem 6.2. *There Exist Infinite Triangular Numbers That Are the sum of Five Triangular Numbers.*

Proof. $((\mathbf{2\xi + 1})^{\mathbf{2}})^{\mathbf{2}} + \mathbf{T_{(3\xi+1)}} = (T_{2\xi} + T_{2\xi+1})^2 + T_{(3\xi+1)}$

$$= T_{(2\xi+1)^2} + T_{[(2\xi+1)^2+1]} + T_{2\xi} + T_{2\xi+1} + T_{\xi}$$

$$= T_{[(2\xi+1)^2+(3\xi+1)]} \cdot [\xi \geq 1]$$ proving the theorem. □

Theorem 6.3. *There Exist Infinite Triangular Numbers That Are the sum of Seven Triangular Numbers.*

Proof. $(((\mathbf{2}\xi + \mathbf{1})^{\mathbf{2}})^{\mathbf{2}})^{\mathbf{2}} + \mathbf{T}_{[(\mathbf{2}\xi+\mathbf{1})^{\mathbf{2}}+(\mathbf{3}\xi+\mathbf{1})]} =$

$T_{[((2\xi+1)^2)^2]} + T_{[((2\xi+1)^2)^2+1]} +$

$T_{(2\xi+1)^2} + T_{[(2\xi+1)^2+1]} + T_{2\xi} + T_{2\xi+1} + T_{\xi}$

$$= T_{[((2\xi+1)^2)^2+(2\xi+1)^2+(3\xi+1)]}. \quad [\xi \geq 1]$$

proving the theorem.

□

Theorem 6.4. *There Exist Infinite Triangular Numbers That Are the sum of Nine Triangular Numbers.*

Proof. $((((\mathbf{2}\xi + \mathbf{1})^{\mathbf{2}})^{\mathbf{2}})^{\mathbf{2}})^{\mathbf{2}} + \mathbf{T}_{[((\mathbf{2}\xi+\mathbf{1})^{\mathbf{2}})^{\mathbf{2}}+(\mathbf{2}\xi+\mathbf{1})^{\mathbf{2}}+(\mathbf{3}\xi+\mathbf{1})]} =$

$T_{[(((2\xi+1)^2)^2)^2]} + T_{[(((2\xi+1)^2)^2)^2+1]} + T_{[((2\xi+1)^2)^2]} +$

$T_{[((2\xi+1)^2)^2]+1]} + T_{(2\xi+1)^2} + T_{[(2\xi+1)^2+1]} + T_{2\xi} + T_{2\xi+1} + T_{\xi}$

$$= T_{[(((2\xi+1)^2)^2)^2+((2\xi+1)^2)^2+(2\xi+1)^2+(3\xi+1)]} \cdot \quad [\xi \geq 1]$$

proving the theorem.

□

Theorem 6.$\tau[\tau \geq 1]$. *There Exist Infinite Triangular Numbers That Are the sum of* $(2\tau+1)$*,* $[\tau \geq 1]$ *Triangular Numbers.*

We continue the Inductive Proof for $[\tau \geq 3]$

$$((((2\xi+1)^2)^2)^2)^{\tau \text{ times}} +$$

$$T_{[(((2\xi+1)^2)^2))))^{(\tau-2)\text{times}} - - - + ((2\xi+1)^2)^2 + (2\xi+1)^2 + (3\xi+1)]} =$$

$$T_{[((((2\xi+1)^2)^2)^2)^{(\tau-1)\text{times}}]} + T_{[((((2\xi+1)^2)^2)^2)^{(\tau-1)\text{times}} + 1]} +$$

$$T_{[((((2\xi+1)^2)^2)^2)^{(\tau-2)\text{times}}]} + T_{[((((2\xi+1)^2)^2)^2)^{(\tau-2)\text{times}} + 1]} +$$

$$T_{[((((2\xi+1)^2)^2)^2)^{(\tau-3)\text{times}}]} + T_{[((((2\xi+1)^2)^2)^2)^{(\tau-3)\text{times}} + 1]} +$$

$$- - - - - - - -$$

$$T_{[(((2\xi+1)^2)^2)]} + T_{[((2\xi+1)^2)^2)+1]} +$$

$$T_{(2\xi+1)^2} + T_{[(2\xi+1)^2+1]} + T_{2\xi} + T_{2\xi+1} + T_{\xi}$$

$$= T_{[((((2\xi+1)^2)^2)^2)))^{(\tau-1)\text{times}} + - - - + ((2\xi+1)^2)^2} + (2\xi+1)^2 + (3\xi+1)] \cdot$$

$$[\xi \geq 1][\tau \geq 3]$$

Simplifying

$$(2\xi+1)^{2^\tau} + T_{[(2\xi+1)^{2^{(\tau-2)}} + - - - + (2\xi+1)^4 + (2\xi+1)^2 + (3\xi+1)]} =$$

$$T_{[(2\xi+1)^{2^{(\tau-1)}}]} + T_{[(2\xi+1)^{2^{(\tau-1)}} + 1]} +$$

$$T_{[(2\xi+1)^{2^{(\tau-2)}}]} + T_{[(2\xi+1)^{2^{(\tau-2)}} + 1]} +$$

$T_{[(2\xi+1)^{2^{(\tau-3)}}]} + T_{[(2\xi+1)^{2^{(\tau-3)}}+1]} +$

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$$T_{[(2\xi+1)^4]} + T_{[(2\xi+1)^4+1]} + T_{(2\xi+1)^2} + T_{[(2\xi+1)^2+1]}$$

$$+T_{2\xi} + T_{2\xi+1} + T_{\xi}$$

$$= T_{[(2\xi+1)^{2^{(\tau-1)}}+---+(2\xi+1)^4+(2\xi+1)^2+(3\xi+1)]} \cdots \quad [\xi \geq 1][\tau \geq 3]$$

proving the theorems.

We have Proved Theorem 1.2 that

If $T = \{T_0, T_1, T_2, T_3, ---T_n, ---, ---\infty\}[T_0 = 0]$ are Triangular Numbers then $(2\lambda)^2 T_n + T_\lambda + \lambda n[\lambda \geq 1, n \geq 0]$ are also Triangular Numbers.

We can utilise this theorem to generate Infinite Inductive Theorems as follows. The self-evident (though a bit cumbersome) inductive steps of the Proofs are omitted.

Theorem 6.5. *There Exist Infinite Triangular Numbers That Are the sum of three Triangular Numbers and a specified constant* $\lambda = 1, 2, 3, ---\infty$ *or Equivalently.*

For each $\lambda = 1, 2, 3, ---, \infty$ *There Exist Three Triangular Numbers, the Sum of which and* λ *is yet another Triangular Number.*

Proof. Substituting $n = 1$ in "$(T_{2\lambda-1} + T_{2\lambda})T_n + T_\lambda + \lambda n = T_{3\lambda n}$" [1.2] we have $T_{2\lambda-1} + T_{2\lambda} + T_\lambda + \lambda = T_{3\lambda} \quad [\lambda \geq 1]$ proving the theorem.

When λ is a Triangular Number we have the Infinite cases when we have Triangular Numbers that are the Sum of Four Triangular Numbers

$$\mathbf{T}_{[\mathbf{n(n+1)-1}]} + \mathbf{T}_{\mathbf{n(n+1)}} + \mathbf{T}_{\frac{[\mathbf{n(n+1)}]}{\mathbf{2}}} + \mathbf{T}_{\mathbf{n}} = T_{\frac{3n(n+1)}{2}} \quad [n \geq 1]$$

proving the theorem. □

Theorem 6.6. *There Exist Infinite Triangular Numbers That Are the sum of Five Triangular Numbers and a specified constant* $\lambda = 1, 2, 3, ---, \infty$ *or Equivalently.*

For each $\lambda = 1, 2, 3, ---, \infty$ *There Exist Five Triangular Numbers, the Sum of which and* λ *is yet another Triangular Number.*

Proof. $((\mathbf{2}\lambda)^{\mathbf{2}})^{\mathbf{2}} + \mathbf{T}_{\mathbf{3\lambda}} = (T_{2\lambda-1} + T_{2\lambda})^2 + T_{3\lambda}$

$$= T_{(2\lambda)^2} + T_{[(2\lambda)^2+1]} + T_{2\lambda-1} + T_{2\lambda} + T_{\lambda} + \lambda$$

$$= T_{[(2\lambda)^2+(3\lambda)]} \cdot \qquad [\lambda \geq 1] \text{ proving the theorem.}$$

When λ is a Triangular Number we have the Infinite cases when we have Triangular Numbers that are the Sum of Six Triangular Numbers

$$T_{[(n(n+1))^2]} + T_{[(n(n+1))^2+1]} + T_{[n(n+1)-1]} + T_{n(n+1)} + T_{\frac{[n(n+1)]}{2}} + T_n$$
$$= T_{[(n(n+1))^2+\frac{3n(n+1)}{2}]} \cdot \quad [n \geq 1]$$

proving the theorem. □

Theorem 6.7. *There Exist Infinite Triangular Numbers That Are the sum of Seven Triangular Numbers and a specified constant* $\lambda = 1, 2, 3, ---, \infty$ *or Equivalently.*

For each $\lambda = 1, 2, 3,$ - - -, ∞ *There Exist Seven Triangular Numbers, the Sum of which and* λ *is yet another Triangular Number.*

Proof.

$$\begin{aligned}(((\mathbf{2\lambda})^{\mathbf{2}})^{\mathbf{2}})^{\mathbf{2}} + \mathbf{T}_{[(\mathbf{2\lambda})^{\mathbf{2}}+(\mathbf{3\lambda})]} &= T_{[((2\lambda)^2)^2]} + T_{[((2\lambda)^2)^2+1]} \\ &\quad + T_{(2\lambda)^2} + T_{[(2\lambda)^2+1]} + T_{2\lambda-1} + T_{2\lambda} + T_{\lambda} + \lambda \\ &= T_{[((2\lambda)^2)^2+(2\lambda)^2+(3\lambda)]} \cdot \quad [\lambda \geq 1]\end{aligned}$$

proving the theorem.

When λ is a Triangular Number we have the Infinite cases when we have Triangular Numbers that are the Sum of Eight Triangular Numbers

$$T_{[((n(n+1))^2)^2]} + T_{[((n(n+1))^2)^2+1]} + T_{[(n(n+1))^2]} +$$

$$T_{[(n(n+1))^2+1]} + T_{[(n(n+1))-1]} + T_{(n(n+1))} + T_{\frac{[n(n+1)]}{2}} + T_n$$

$$= T_{[((n(n+1))^2)^2+(n(n+1))^2+\frac{3n(n+1)}{2}]} \cdot \quad [n \geq 1]$$

proving the theorem. □

Theorem 6.8. *There Exist Infinite Triangular Numbers That Are the sum of Nine Triangular Numbers and a specified constant* $\lambda = 1, 2, 3,$ - - -, ∞ *or Equivalently.*

For each $\lambda = 1, 2, 3,$ - - -, ∞ *There Exist Nine Triangular Numbers, the Sum of which and* λ *is yet another Triangular Number.*

Proof. $((((\mathbf{2}\lambda)^{\mathbf{2}})^{\mathbf{2}})^{\mathbf{2}})^{\mathbf{2}} + \mathbf{T}_{[((\mathbf{2}\lambda)^{\mathbf{2}})^{\mathbf{2}}+(\mathbf{2}\lambda)^{\mathbf{2}}+(\mathbf{3}\lambda)]} =$

$T_{[(((2\lambda)^2)^2)^2]} + T_{[(((2\lambda)^2)^2)^2+1]} + T_{[((2\lambda)^2)^2]} + T_{[((2\lambda)^2)^2+1]} +$

$T_{(2\lambda)^2} + T_{[(2\lambda)^2+1]} + T_{2\lambda-1} + T_{2\lambda} + T_{\lambda} + \lambda$

$= T_{[(((2\lambda)^2)^2)^2+((2\lambda)^2)^2+(2\lambda)^2+(3\lambda)]}. \quad [\lambda \geq 1]$

proving the theorem.

When λ is a Triangular Number we have the Infinite cases when we have Triangular Numbers that are the Sum of Ten Triangular Numbers

$T_{[(((n(n+1))^2)^2)^2]} + T_{[(((n(n+1))^2)^2)^2+1]}$

$T_{[((n(n+1))^2)^2]} + T_{[((n(n+1))^2)^2+1]}$

$T_{[(n(n+1))^2]} + T_{[(n(n+1))^2+1]} + T_{[n(n+1)-1]} + T_{n(n+1)} + T_{\frac{[n(n+1)]}{2}} + T_n$

$= T_{[(((n(n+1))^2)^2)^2+((n(n+1))^2)^2+(n(n+1))^2+\frac{3n(n+1)}{2}]}. \quad [n \geq 1]$

proving the theorem. □

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Theorem 6. $\psi[\psi \geq 1]$. *There Exist Infinite Triangular Numbers That Are the sum of* $(2\psi + 1)$ $[\psi \geq 1]$ *Triangular Numbers and a specified constant* $\lambda = 1, 2, 3, ---, \infty$ *Or Equivalently.*

For each $\lambda = 1, 2, 3, ---, \infty$ *There Exist* $(2\psi + 1)[\psi \geq 1]$ *Triangular Numbers, the Sum of which and* λ *is yet another Triangular Number.*

We continue the Inductive Proof for $[\psi \geq 3]$

$((((2\lambda)^2)^2)^2))))^{\psi \text{ times}}+$

$T_{[(((2\lambda)^2)^2))))^{(\psi-2)\text{ times}}---+((2\lambda)^2)^2(2\lambda)^2+(3\lambda)]} =$

$T_{[((((2\lambda)^2)^2)^2))^{(\psi-1)\text{ times}}]} + T_{[((((2\lambda)^2)^2)^2))^{(\psi-1)\text{ times}}+1]}+$

$T_{[((((2\lambda)^2)^2)^2))^{(\psi-2)\text{ times}}]} + T_{[((((2\lambda)^2)^2)^2))^{(\psi-2)\text{ times}}+1]}+$

$T_{[((((2\lambda)^2)^2)^2))^{(\psi-3)\text{ times}}]} + T_{[((((2\lambda)^2)^2)^2))^{(\psi-3)\text{ times}}+1]}+$

$- - - - - - - --$

$T_{[((2\lambda)^2)^2]} + T_{[((2\lambda)^2)^2+1]} + T_{(2\lambda)^2} + T_{[(2\lambda)^2+1]} + T_{2\lambda-1} + T_{2\lambda} + T_\lambda + \lambda$

$= T_{[((((2\lambda)^2)^2)^2)^{(\psi-1)\text{ times}}+---+((2\lambda)^2)^2+(2\lambda)^2+(3\lambda)]}. \quad [\lambda \geq 1] \quad [\psi \geq 3]$

Simplifying

$(2\lambda)^{2^\psi} + T_{[(2\lambda)^{2^{\psi-2}}+---+(2\lambda)^4+(2\lambda)^2+(3\lambda)]} =$

$T_{[(2\lambda)^{2^{(\psi-1)}}]} + T_{[(2\lambda)^{2^{(\psi-1)}}+1]}+$

$T_{[(2\lambda)^{2^{(\psi-2)}}]} + T_{[(2\lambda)^{2^{(\psi-2)}}+1]}+$

$T_{[(2\lambda)^{2^{(\psi-3)}}]} + T_{[(2\lambda)^{2^{(\psi-3)}}+1]}+$

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$T_{[(2\lambda)^4]} + T_{[(2\lambda)^4+1]} + T_{(2\lambda)^2} + T_{[(2\lambda)^2+1]} + T_{2\lambda-1} + T_{2\lambda} + T_\lambda + \lambda$

$= T_{[(2\lambda)^{2^{(\psi-1)}}+---+(2\lambda)^4+(2\lambda)^2+(3\lambda)]}. \qquad [\lambda \geq 1][\psi \geq 3]$

proving the theorems.

When λ is a Triangular Number we have the Infinite cases when we have Triangular Numbers that are the Sum of $(2\psi + 2) = 2(\psi + 1)[\psi \geq 1]$ Triangular Numbers.

We continue the Inductive Proof for $[\psi \geq 3]$

$T_{[(n(n+1))^{2(\psi-1)}]} + T_{[(n(n+1))^{2(\psi-1)}+1]} +$

$T_{[(n(n+1))^{2(\psi-2)}]} + T_{[(n(n+1))^{2(\psi-2)}+1]} +$

$T_{[(n(n+1))^{2(\psi-3)}]} + T_{[(n(n+1))^{2(\psi-3)}+1]} +$

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$T_{[(n(n+1))^4]} + T_{[(n(n+1))^4+1]} +$

$T_{[(n(n+1))^2]} + T_{[(n(n+1))^2+1]} + T_{[n(n+1)^2-1]} + T_{n(n+1)} + T_{\frac{[n(n+1)]}{2}} + T_n$

$= T_{[(n(n+1))^{2(\psi-1)}+ - - - +(n(n+1))^4+(n(n+1))^2+\frac{3n(n+1)}{2}]}. \quad [n \geq 1]$

proving the theorems.

This Proves the General theorem

Theorem 6.9. *For any Natural Number $N \geq 2$, there Exist Infinite Triangular Numbers that are the Sum of N Triangular Numbers each.*

Proof. For $N = 2$ we have the standard result. (Ref. 6)

For $N > 2$, we have proved the result.

For $N = 3, 5, 7,$ - - - and $N = 4, 6, 8,$ - - - in Theorems 6) and Theorems 6a) respectively. This Completes the Proof. □

7 Theorems on two special classes of Triangular Numbers

Theorem 7.1. *If* $x = \dfrac{n(n+2\alpha+1)}{2}+1$, $y = n+1$, $z = \dfrac{n(n+2\alpha+1)}{2}$

$$T_x = T_y + T_z + (\alpha - 1)n = T_y + T_z + (\alpha - 1)(y - 1)$$

$$[\alpha = 0, 1, 2, \text{---}, \alpha] \qquad [n = 0, 1, 2, \text{---}, \alpha]$$

Proof. If $x = \dfrac{n(n+2\alpha+1)}{2} + 1$, $y = n + 1$, $z = \dfrac{n(n+2\alpha+1)}{2}$

7–1–1) For, $\alpha = 0$ we can prove by inducting on "n"

$$x = \frac{n(n+1)}{2} + 1, \quad y = n + 1, \quad z = \frac{n(n+1)}{2}$$
$$T_x = T_y + T_z - n = T_y + T_z - (y - 1) \quad [n = 0, 1, 2, \text{---}, \infty]$$

7–1–2) For, $\alpha = 1$ we can prove by inducting on "n"

$$x = \frac{n(n+3)}{2} + 1, \quad y = n + 1, \quad z = \frac{n(n+3)}{2}$$
$$T_x = T_y + T_z \quad [n = 0, 1, 2, \text{---}, \infty]$$

which is the standard result that states that there exist infinite Triangular Numbers that are the sum of two other Triangular Numbers. [Ref. 6]

7–1–3) For, $\alpha = 2$ we can prove by inducting on "n"

$$x = \frac{n(n+5)}{2} + 1, \quad y = n + 1, \quad z = \frac{n(n+5)}{2}$$
$$T_x = T_y + T_z + n = T_y + T_z + (y - 1) \quad [n = 0, 1, 2, \text{---}, \infty]$$

7–1–4) For, $\alpha = 3$ we can prove by inducting on "n"

$$x = \frac{n(n+7)}{2} + 1, \quad y = n+1, \quad z = \frac{n(n+7)}{2}$$
$$T_x = T_y + T_z + n = T_y + T_z + (y-1) \quad [n = 0, 1, 2, \text{- - -}, \infty]$$

7–1–5) For, $\alpha = 4$ we can prove by inducting on "n"

$$x = \frac{n(n+9)}{2} + 1, \quad y = n+1, \quad z = \frac{n(n+9)}{2}$$
$$T_x = T_y + T_z + n = T_y + T_z + (y-1) \quad [n = 0, 1, 2, \text{- - -}, \infty]$$

7–1–6) For, $\alpha = 5$ we can prove by inducting on "n"

$$x = \frac{n(n+11)}{2} + 1, \quad y = n+1, \quad z = \frac{n(n+11)}{2}$$
$$T_x = T_y + T_z + n = T_y + T_z + (y-1) \quad [n = 0, 1, 2, \text{- - -}, \infty]$$

7–1–k+1) For, $\alpha = k$ we can prove by inducting on "n"

$$x = \frac{n(n+2k+1)}{2} + 1, \quad y = n+1, z = \frac{n(n+2k+1)}{2}$$
$$T_x = T_y + T_z + (k-1)n = T_y + T_z + (k-1)(y-1)$$
$$[k = 0, 1, 2, \cdots, \infty] \; [n = 0, 1, 2, \cdots, \infty]$$

7–1–k+2) For, $\alpha = k+1$ we can prove by inducting on "n"

$$x = \frac{n(n+2(k+1)+1)}{2} + 1, \quad y = n+1, \; z = \frac{n(n+2(k+1)+1)}{2}$$

$$T_x = T_y + T_z + (k)n = T_y + T_z + (k)(y-1)$$
$$[k = 0, 1, 2, \cdots, \infty]\ [n = 0, 1, 2, \cdots, \infty]$$

thus Completing the Induction on "α" proving the theorem that, if

$$x = \frac{n(n+2\alpha+1)}{2} + 1, \quad y = n+1,\ z = \frac{n(n+2\alpha+1)}{2}$$
$$T_x = T_y + T_z + (\alpha-1)n = T_y + T_z + (\alpha-1)(y-1)$$
$$[\alpha = 0, 1, 2, \cdots, \infty]$$

$[n = 0, 1, 2, \cdots, \infty]$ □

Theorem 7.2. *If*

$$x = \frac{(n+1)(n+2\beta)}{2} + 1, \quad y = n+1, z = \frac{(n+1)(n+2\beta)}{2}$$
$$T_x = T_y + T_z + \beta + (\beta-1)n = T_y + T_z + \beta + (\beta-1)(y-1)$$
$$[\beta = 1, 2, 3, \cdots, \infty]\ [n = 0, 1, 2, \cdots, \infty]$$

Proof. If

$$x = \frac{(n+1)(n+2\beta)}{2} + 1, \quad y = n+1, z = \frac{(n+1)(n+2\beta)}{2}.$$

7–2–1) For, $\beta = 1$ we can prove by inducting on "n"

$$x = \frac{(n+1)(n+2)}{2} + 1, \quad y = n+1, z = \frac{(n+1)(n+2)}{2}$$
$$T_x = T_y + T_z + 1 \quad [n = 0, 1, 2, \text{---}, \infty]$$

7–2–2) For, $\beta = 2$ we can prove by inducting on "n"

$$x = \frac{(n+1)(n+4)}{2} + 1, \quad y = n+1, z = \frac{(n+1)(n+4)}{2}$$
$$T_x = T_y + T_z + 2 + n = T_y + T_z + 2 + (y-1) \quad [n = 0, 1, 2, \text{---}, \infty]$$

7–2–3) For, $\beta = 3$ we can prove by inducting on "n"

$$x = \frac{(n+1)(n+6)}{2} + 1, \quad y = n+1, z = \frac{(n+1)(n+6)}{2}$$
$$T_x = T_y + T_z + 3 + 2n = T_y + T_z + 3 + 2(y-1) \quad [n = 0, 1, 2, \text{---}, \infty]$$

7–2–4) For, $\beta = 4$ we can prove by inducting on "n"

$$x = \frac{(n+1)(n+8)}{2} + 1, \quad y = n+1, z = \frac{(n+1)(n+8)}{2}$$
$$T_x = T_y + T_z + 4 + 3n = T_y + T_z + 4 + 3(y-1) \ [n = 0, 1, 2, \cdots, \infty]$$

7–2–5) For, $\beta = 5$ we can prove by inducting on "n"

$$x = \frac{(n+1)(n+10)}{2} + 1, \quad y = n+1, z = \frac{(n+1)(n+10)}{2}$$
$$T_x = T_y + T_z + 5 + 4n = T_y + T_z + 5 + 4(y-1) \quad [n = 0, 1, 2, \cdots, \infty]$$

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7–2–k) For, $\beta = k$ we can prove by inducting on "n"

$$x = \frac{(n+1)(n+2k)}{2} + 1, \quad y = n+1, z = \frac{(n+1)(n+2k)}{2}$$
$$T_x = T_y + T_z + k + (k-1)n = T_y + T_z + k + (k-1)(y-1)$$
$$[k = 1, 2, 3, \cdots, \infty] \; [n = 0, 1, 2, \cdots, \infty]$$

7–2–k) For, $\beta = k+1$ we can prove by inducting on "n"

$$x = \frac{(n+1)(n+2(k+1))}{2} + 1, \quad y = n+1, z = \frac{(n+1)(n+2(k+1))}{2}$$
$$T_x = T_y + T_z + (k+1) + (k)n = T_y + T_z + (k+1) + (k)(y-1)$$
$$[k = 1, 2, 3, \cdots, \infty] \; [n = 0, 1, 2, \cdots, \infty]$$

thus completing the induction on "β " proving the theorem that

If

$$x = \frac{(n+1)(n+2\beta)}{2} + 1, \; y = n+1, z = \frac{(n+1)(n+2\beta)}{2}$$
$$T_x = T_y + T_z + \beta + (\beta-1)n = T_y + T_z + \beta + (\beta-1)(y-1)$$
$$[\beta = 1, 2, 3 \text{---}, \infty] \; [n = 0, 1, 2, \text{---}, \infty]$$

□

We come to the end of this paper. The Super-Sums of each Sub-Class of Triangular Numbers upto any Level and other interesting things are possible. [Ref. 7,8,9.]

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